PHYS5150 — PLASMA PHYSICS

LECTURE 6 - ADIABATIC INVARIANTS

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1 ADIABATIC INVARIANTS

The presence of adiabatic invariants is actually a common phenomenon, which has been studied extensively in classical mechanics. Here we follow *Landau & Lifschitz* and consider a one-dimensional finite motion, where λ is a parameter describing a very slow change of the system. Here, slow means slow compared to the period T of the cyclic motion, i.e. $T\dot{\lambda} \ll \lambda$. Now, because λ is slowly changing, so is the energy E of the system, where $\dot{E} \sim \dot{\lambda}$. This implies that the change of energy is a function of λ, from what follows that there is a combination of E and λ, a so-called *adiabatic invariant*, which remains constant.

Now let $H(p,q;\lambda)$ be the Hamiltonian of such a system, where again λ is the parameter characterizing the slow change. Then,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}.$$

Now we average over one cycle T and assume that $\dot{\lambda}$ does not change on this time scale

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{\mathrm{d}\lambda}{\mathrm{d}t}\overline{\frac{\partial H}{\partial \lambda}}.$$

Now,

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^T \frac{\partial H}{\partial \lambda}\,\mathrm{d}t,$$

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and using that $\dot{q} = \frac{\partial H}{\partial p}$ we obtain

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T} \oint \frac{\partial H}{\partial \lambda} \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q.$$

By further noting that

$$T = \int_0^T \mathrm{d}t = \oint \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q$$

we get

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{\oint \frac{\partial H}{\partial \lambda} \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q}{\oint \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q},$$

and thus

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{\mathrm{d}\lambda}{\mathrm{d}t} \frac{\oint \frac{\partial H}{\partial \lambda} \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q}{\oint \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q}.$$

We have assumed that λ is constant along the integration path, which implies that $E = H(p,q;\lambda)$ is constant as well. Differentiating H with respect to λ gives

$$0 = \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial \lambda},$$

and thus

$$\frac{\partial H}{\partial \lambda} \left(\frac{\partial H}{\partial p} \right)^{-1} = -\frac{\partial p}{\partial \lambda}.$$

After substituting this expression into our expression for the change of the mean energy we get

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = -\frac{\mathrm{d}\lambda}{\mathrm{d}t} \frac{\oint \frac{\partial p}{\partial \lambda} \mathrm{d}q}{\oint \frac{\partial p}{\partial E} \mathrm{d}q},$$

or

$$0 = \oint \left(\frac{\partial p}{\partial E} \frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial p}{\partial \lambda} \frac{\mathrm{d}\lambda}{\mathrm{d}t} \right) \mathrm{d}q = \frac{\mathrm{d}}{\mathrm{d}t} \oint p \, \mathrm{d}q.$$

This result implies that the *adiabatic invariant*

$$\boxed{I = \frac{1}{2\pi} \oint p \, \mathrm{d}q} \tag{1}$$

remains constant even when the parameter λ is changing slowly. I is actually the area enclosed by periodic path of the system in the phase space.

1.1 *Example: Harmonic Oscillator*

As an example lets us consider a harmonic oscillator, which has the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2.$$

The system's path describes an ellipse with the semi-major axises $\sqrt{2mE}$ and $\sqrt{2E/m\omega^2}$, and the area

$$A = 2\pi\sqrt{2mE}\sqrt{2E/m\omega^2} = 2\pi\frac{E}{\omega}.$$

This implies that the oscillator has an adiabatic invariant

$$\boxed{I_{osc} = \frac{E}{\omega},} \tag{2}$$

which is conserved even when the oscillator's mass or k varies.

2 MAGNETIC MOMENT AS A CONSTANT OF MOTION

We now investigate the guiding center motion of a charged particle along an inhomogeneous magnetic field. We assume that the field is axially symmetric (i.e. $\mathbf{B} = (B_\rho, B_\phi, B_z)$ with $\partial_\phi \mathbf{B} = 0$), where the symmetry axis z is aligned with the field gradient $\nabla\mathbf{B} = \partial_z B_z$. We only consider particle motions close to the symmetry axis where we can safely ignore the dependence of $\partial_z B_z$ on the radial distance ρ.

From Gauss' law $\nabla \cdot \mathbf{B} = 0$ follows that

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0,$$

and after performing the integration with respect to ρ

$$B_\rho = -\frac{1}{2}\left(\frac{\partial B_z}{\partial z}\right)\rho. \tag{3}$$

Note that this relation is only valid close to the symmetry axis because we assumed that $\frac{\partial B_z}{\partial z} \neq f(\rho)$. The particle's motion parallel to the symmetry axis is given by

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = F_z = q\left(v_x B_y - v_y B_x\right), \tag{4}$$

where the field components B_x and B_y are given by Eq. (3)

$$B_x = -\frac{1}{2}q\left(\frac{\partial B_z}{\partial z}\right)x, \tag{5}$$

$$B_y = -\frac{1}{2}q\left(\frac{\partial B_z}{\partial z}\right)y. \tag{6}$$

With that we get

$$F_z = -\frac{1}{2}q\frac{\partial B_z}{\partial z}\left(v_x y - v_y x\right). \tag{7}$$

Let us now assume that $\frac{\partial B_z}{\partial z}$ is small, so that the motion in the x-y plane will be circular

$$x = \rho_c \sin\omega_c t$$

$$y = \rho_c \cos\omega_c t \frac{q}{|q|}.$$

The $q/|q|$ term in the expression for y accounts for the direction of the gyro motion. The corresponding velocity components are then

$$v_x = \omega_c \rho_c \cos\omega_c t,$$

$$v_y = -\frac{q}{|q|}\omega_c \rho_c \sin\omega_c t,$$

and thus

$$F_z = -\frac{\partial B_z}{\partial z}\left(\frac{|q|}{2}\omega_c \rho_c^2\right).$$

Recall that the *magnetic moment* is

$$\mu = \frac{m\mathbf{v}_\perp^2}{2B} = \frac{T_\perp}{B},$$

or after expressing $\mathbf{v}_\perp$ by ω_c and ρ_c

$\omega_c =; \rho_c =$

$$\mu = \left(\frac{|q|}{2}\omega_c \rho_c^2\right),$$

and hence

$$\boxed{F_z = -\frac{\partial B_z}{\partial z}\mu.} \tag{8}$$

This result implies that the particle is repelled from strong magnetic field regions.

Now we have a closer look at the particle's azimuthal motion in the x-y plane. Here the force acting on the particle is

$$F_\phi = q v_z B_\rho, \tag{9}$$

from which follows that the rate of change of the kinetic energy of the motion in this plane is

$$\frac{\mathrm{d}T_\perp}{\mathrm{d}t} = v_\phi q v_z B_\rho.$$

After using Eq. (3) and replacing v_ϕ by $-q/|q|\mathbf{v}_\perp$ we find that

$$\frac{\mathrm{d}T_\perp}{\mathrm{d}t} = |q|\mathbf{v}_\perp v_z \frac{\partial B_z}{\partial z}\frac{\rho}{2}.$$

Note that while the total kinetic energy T is conserved, $T_\perp$ is not constant. After replacing ρ with ρ_c we get

$$\boxed{\dot{T}_\perp = \frac{T_\perp v_z}{B}\frac{\partial B_z}{\partial z}.} \tag{10}$$

Finally, knowledge of $\dot{T}_\perp$ enables us to derive the rate of change of the magnetic moment

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T_\perp}{B}\right) = \frac{1}{B}\dot{T}_\perp - \frac{T_\perp}{B^2}\dot{B}.$$

Using that $\dot{B} = v_z \partial_z B_z$ yields

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{1}{B}\dot{T}_\perp - \frac{T_\perp}{B^2} v_z \frac{\partial B_z}{\partial z}$$

and after inserting Eq. (10)

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{T_\perp}{B^2} v_z \frac{\partial B_z}{\partial z} - \frac{T_\perp}{B^2} v_z \frac{\partial B_z}{\partial z} = 0.$$

The magnetic moment μ is a constant of motion for a $\mathbf{B}\|\nabla\mathbf{B}$ field configuration. Such a field configuration constitute a *magnetic mirror* – a particle moving into the strong field region will eventually reflected back into the weak field domain.