# PHYS5150 — PLASMA PHYSICS

## LECTURE 5 - SINGLE PARTICLE MOTION IN AN UNIFORM B FIELD

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*Single particle motion in an uniform B field*

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### PLASMA PARAMETERS

COMPOSITION: ions and electrons

NUMBER DENSITY: ions and electrons in laboratory plasmas $\sim 10^8 - 10^{14}\text{cm}^{-3}$

TEMPERATURE: measured in electron Volts ($e$V), $1e\text{V} = 11,600\text{K}$

$e \cdot U = \frac{m}{2}v^2 = \text{k}_\text{B}T = e \cdot 1\text{V}$
$1e\text{V} = 1.602 \cdot 10^{-19}\text{C} \cdot \text{J/C}$
$1e\text{V} = 1.602 \cdot 10^{-19}\text{J}$

DISTANCE SCALE: Debye length $\lambda_D$

TIME SCALE: plasma frequency $\omega_p = 2\pi f_p$

VELOCITY SCALE: thermal velocity $v_{th} = \sqrt{\frac{8kT}{\pi m}}$

### 1 SINGLE PARTICLE MOTION IN A UNIFORM B FIELD

Before we deal with the really messy stuff, it is beneficial to study the motion of single charged particles in uniform electric and magnetic fields. As a first step let's investigate the case of a charged grain moving in an uniform magnetic field.

You will show in your homework assignment that if only a Lorentz force

$$\mathbf{F}_L = q\left(\mathbf{v} \times \mathbf{B}\right)$$

acts on a charged particle, its kinetic energy $T = \frac{1}{2}m\mathbf{v}^2$ is an integral of motion. We now split $\mathbf{v}$ into its components parallel and orthogonal to the magnetic field:

$$\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$$

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and similarly

$$T = \frac{1}{2}m\mathbf{v}_{\|}^2 + \frac{1}{2}m\mathbf{v}_{\perp}^2 = T_{\|} + T_{\perp}.$$

Because of $\mathbf{v} \times \mathbf{B} = \mathbf{v}_{\|} \times \mathbf{B} + \cancel{\mathbf{v}_{\perp} \times \mathbf{B}} = \mathbf{v}_{\|} \times \mathbf{B}$, $T_{\|}$ is an integral of motion, and $T_{\perp}$ is an integral of motion, too.

Then, the equation of motion for the component $\mathbf{v}_{\perp}$

$$m\frac{\mathrm{d}\mathbf{v}_{\perp}}{\mathrm{d}t} = q\mathbf{v}_{\perp}\mathbf{B} = m\frac{\mathbf{v}_{\perp}^2}{\rho_c}$$

describes a circular motion around the so-called guiding center. The radius

$$\boxed{\rho_c = \frac{mv_{\perp}}{|q|B}} \tag{1}$$

is the *cyclotron* or *Lamor radius*. The angular frequency of the cyclotron motion

$$\boxed{\omega_c = \frac{v_{\perp}}{\rho_c} = \frac{|q|B}{m}} \tag{2}$$

is called the *cyclotron* or *Lamor frequency*.

Note that the gyromotion of a charge constitutes a current loop

$$j = \frac{q}{\Delta t} = q\frac{\omega_c}{2\pi} = \frac{q}{2\pi}\frac{qB}{m} = \frac{q^2}{2\pi}\frac{B}{m}.$$

In this case, there is a *magnetic moment*

$$\mu = \text{area} \cdot \text{current}.$$

The area enclosed by the loop current is

$$A_{loop} = \pi\rho_c^2 = \pi\frac{m^2v_{\perp}^2}{|q|^2B^2},$$

and thus

$$\mu = j \cdot A_{loop} = \frac{q^2}{2\pi}\frac{B}{m}\pi\frac{m^2v_{\perp}^2}{|q|^2B^2} = \frac{mv_{\perp}^2}{2B},$$

and finally

$$\boxed{\mu = \frac{T_{\perp}}{B}}. \tag{3}$$

Note that for both, the electrons and the ions, the direction of $\mu$ is opposite to the applied magnetic field $\mathbf{B}$. This means that $\mu$ resulting from the plasma particles' gyromotion weakens the applied field – the plasma is *diamagnetic*.

## 2 UNIFORM B AND E FIELDS

### 2.1 *E field parallel to B*

An E field parallel to $\mathbf{B}$ would only affect $\mathbf{v}_{\|}$ and result in a guiding center motion parallel to $\mathbf{B}$.

### 2.2 *E field perpendicular to B*

This case is more interesting than $\mathbf{E}\|\mathbf{B}$ and will lead us to new insights. Here, the equations of motion are

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\|} = 0$$

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\perp} = q\left(\mathbf{E}+\mathbf{v}_{\perp}\times\mathbf{B}\right).$$

We now transform the equations into an inertial frame moving at constant speed $v_E$ perpendicular to $\mathbf{B}$. The fields in the new reference system are then

$$\mathbf{E}^{'} = \mathbf{E}+\mathbf{v}_E\times\mathbf{B}$$

$$\mathbf{B}^{'} = \mathbf{B},$$

the velocity components are

$$\mathbf{v}_{\|}^{'} = \mathbf{v}_{\|}$$

$$\mathbf{v}_{\perp}^{'} = -\mathbf{v}_E+\mathbf{v}_{\perp},$$

and the new equation of motion for $\mathbf{v}_{\perp}$ is

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\perp}^{'} = q\left(\mathbf{E}^{'}+\mathbf{v}_{\perp}^{'}\times\mathbf{B}^{'}\right).$$

We now choose $\mathbf{v}_E$ such that $\mathbf{E}^{'}$ vanishes, i.e.

$$\mathbf{E}^{'} = \mathbf{E}+\mathbf{v}_E\times\mathbf{B} = 0 \quad \Big|\times\mathbf{B}$$

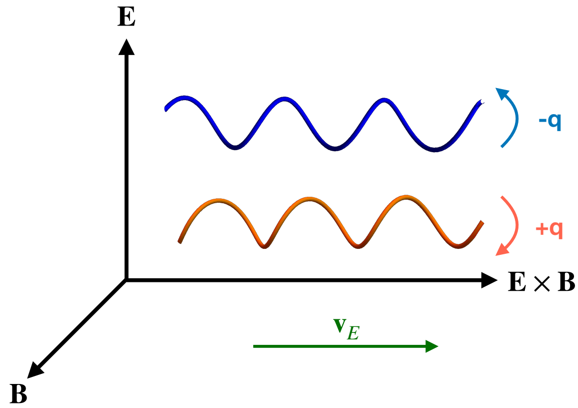
$$= \mathbf{E}\times\mathbf{B}+\left(\mathbf{v}_E\times\mathbf{B}\right)\times\mathbf{B} = \mathbf{E}\times\mathbf{B}+\cancel{\left(\mathbf{v}_E\cdot\mathbf{B}\right)\mathbf{B}}-\left(\mathbf{B}\cdot\mathbf{B}\right)\mathbf{v}_E,$$

*Remember that* $(A\times B)\times C = (A\cdot C)B-(B\cdot C)A$

and finally

$$\boxed{\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2}}. \tag{4}$$

This is the velocity of the particle's guiding center drift caused by a uniform electric field perpendicular to $\mathbf{B}$.




In the prime system the particle performs a simple gyromotion because its equation of motion is simply

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_\perp^{'} = q\left(\cancel{\mathbf{E}'} + \mathbf{v}_\perp^{'}\times\mathbf{B}^{'}\right) = q\left(\mathbf{v}_\perp^{'}\times\mathbf{B}^{'}\right),$$

while in its initial system drifts the guiding center of the gyrating particle in $\mathbf{E}\times\mathbf{B}$ direction with the speed $v_E$.

The $\mathbf{E}\times\mathbf{B}$ drift has some remarkable properties. All particles *drift at the same speed* regardless of their charge, temperature, and mass. Furthermore, the drift motion *does not constitute a current*. Note also that in plasma physics the frame of rest is the one in which the electric field vanishes.

## 3 MOTION IN AN NONUNIFORM B FIELD

### 3.1 $\nabla\mathbf{B}$ *Drift*

We are interested in what happens when the particle moves in a nonuniform magnetic field. Lets assume that the B filed is aligned with the z axis, i.e. $\mathbf{B} = (0,0,B_z)$, and that the density of the magnetic field lines increases in x direction, i.e. $\nabla\mathbf{B}\|\mathbf{x}$. We would expect a drift in y direction, because the radius of the gyromotion increases with decreasing magnetic field strength.

We now want to determine the average drift velocity due to the field gradient. Because the motion in x-direction is periodic

$$\oint F_x\,\mathrm{d}t = q\oint v_y B_z\,\mathrm{d}t = 0.$$

We assume the field gradient to be small, which allows us to expand $B_z$ around its guiding center

$$B_z(x) \approx B_z(x_0) + \frac{\partial B_z}{\partial x}(x - x_0)$$

and get

$$0 = \oint \left\{ B_z(x_0) + \frac{\partial B_z}{\partial x}(x - x_0) \right\} v_y \, \mathrm{d}t$$

$$= B_z(x_0) \underbrace{\oint v_y \, \mathrm{d}t}_{\Delta y} + \frac{\partial B_z}{\partial x} \oint (x - x_0) v_y \, \mathrm{d}t.$$

We now make use of that

$$\oint (x - x_0) v_y \, \mathrm{d}t = \oint (x - x_0) \frac{\mathrm{d}y}{\mathrm{d}t} \, \mathrm{d}t = \oint (x - x_0) \, \mathrm{d}y$$

is approximately the area of a circle with radius $\rho_c$, and the second integral becomes

$$\oint (x - x_0) v_y \, \mathrm{d}t \approx -\frac{q}{|q|} \pi \rho_c^2.$$

Thus

$$0 = B_z \Delta y - \frac{\partial B_z}{\partial x} \frac{q}{|q|} \pi \rho_c^2$$

$$\Delta y = \frac{\frac{\partial B_z}{\partial x}}{B_z} \frac{q}{|q|} \pi \rho_c^2.$$

During one gyration cycle $\Delta t = \frac{2\pi}{\omega_c}$ the particle drifts by $\Delta y$ in y direction, which provides us with the drift speed

$$v_G = \frac{\Delta y}{\Delta t} = \frac{\partial B_z}{\partial x} \frac{1}{B_z} \frac{q}{|q|} \frac{1}{2} \omega_c \rho_c^2 = \frac{T_\perp}{q B_z} \left[ \frac{1}{B_z} \frac{\partial B_z}{\partial x} \right],$$

where we have used that

$$T_\perp = \frac{m}{2} \omega_c^2 \rho_c^2.$$

The general expression for the gradient B drift velocity is

$$\boxed{\mathbf{v}_G = \frac{T_\perp}{qB} \left[ \frac{\hat{\mathbf{B}} \times \nabla \mathbf{B}}{B} \right]}. \tag{5}$$

The direction of the grad B drift is in opposite direction for positive and negative charges and *causes therefore a current*.

### 3.2 *Curvature Drift*

A charged particle moving along a curved magnetic field line will experience a centrifugal force

$$F_C = m\frac{\mathbf{v}_{\parallel}^2}{R_C},$$

where $R_C$ is the field curvature. This leads to a *curvature drift*

$$\mathbf{v}_C = -m\frac{\mathbf{v}_{\parallel}^2}{R_C^2}\left[\frac{\hat{\mathbf{R}}_C \times \hat{\mathbf{B}}}{qB^2}\right]$$

or after introducing the kinetic energy of the parallel motion $T_{\parallel} = \frac{1}{2}m\mathbf{v}_{\parallel}^2$

$$\boxed{\mathbf{v}_C = 2\frac{T_{\parallel}}{qB}\left[\frac{\hat{\mathbf{B}} \times \hat{\mathbf{R}}_C}{R_C}\right].} \tag{6}$$

The direction of the curvature drift is in opposite direction for positive and negative charges and *causes therefore a current*.