PHYS5150 — PLASMA PHYSICS

LECTURE 23 - ELECTROSTATIC WAVES IN COLD MAGNETIZED PLASMAS II

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1 ELECTROSTATIC WAVES IN COLD MAGNETIZED PLASMAS (CNTD.)

In the previous lecture we have found the dispersion relation for electrostatic waves propagating through a cold plasma

$$\boxed{\omega^2 = \sum_{s=i,e} \frac{\omega_{p,s}^2}{k^2}\left[k_z^2 + \frac{k_x^2}{1-\frac{\omega_{c,s}^2}{\omega^2}}\right].} \tag{1}$$

We have already discussed the trivial cases of $\mathbf{B}=0$ and $\mathbf{B}\|\mathbf{k}$, as well as the case of a strongly magnetized plasma. We now consider the more complicated case of $\mathbf{B}\perp\mathbf{k}$.

1.1 *Dispersion relation for* $\mathbf{B}\perp\mathbf{k}$

After setting $k_z = 0$ the dispersion relation simplifies to

$$\omega^2 = \frac{\omega_{pe}^2}{\cancel{k_x^2}}\frac{\cancel{k_x^2}}{1-\frac{\omega_{ce}^2}{\omega^2}} + \frac{\omega_{pi}^2}{\cancel{k_x^2}}\frac{\cancel{k_x^2}}{1-\frac{\omega_{ci}^2}{\omega^2}}$$

$$\omega^2 = \omega_{pe}^2\frac{\omega^2}{\omega^2-\omega_{ce}} + \omega_{pi}^2\frac{\omega^2}{\omega^2-\omega_{ci}}.$$

The resulting dispersion relation has three solutions at $\omega \approx \omega_{ce}$, $\omega \approx \omega_{ci}$, and $\omega_{ci} < \omega < \omega_{ce}$, which are fundamentally different and need to be investigated separately.

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1.1.1 Root near $\omega \approx \omega_{ce} \gg \omega_{ci}$

The frequency ω_{uh} of the resulting *upper hybrid wave*

$$\boxed{\omega_{uh}^2 = \omega_{pe}^2 + \omega_{ce}^2} \tag{2}$$

is called the *upper hybrid frequency*. They are called such because at ω_{uh} the plasma and cyclotron properties of electrons mix.

1.1.2 Root near $\omega \approx \omega_{ci} \ll \omega_{ce}$

The dispersion relation for this solution is

$$\omega^2 = -\frac{\omega^2 \omega_{pe}^2}{\omega_{ce}} + \frac{\omega^2 \omega_{pi}^2}{\omega^2 - \omega_{ci}},$$

or

$$\frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}} = 1 + \frac{\omega_{pe}^2}{\omega_{ce}},$$

and finally

$$\omega^2 = \omega_{ci}^2 + \frac{\omega_{pi}^2}{1 + \frac{\omega_{pe}^2}{\omega_{ce}}}.$$

This is the general solution for $\omega \approx \omega_{ci}$. For strongly magnetized plasmas under lab conditions it is possible to enforce that $\omega_{ce}^2 \gg \omega_{pe}^2$. Under such conditions one can observe *electrostatic cyclotron waves* propagating with the frequency

$$\boxed{\omega^2 = \omega_{pi}^2 + \omega_{ci}^2.} \tag{3}$$

1.1.3 Waves with frequencies between ω_{ci} and ω_{ce}

We first introduce the angle θ between **k** and **B** and rewrite the general dispersion relation as

$$\omega^2 = \sum_{s=i,e} \omega_{p,s}^2 \cos^2\theta + \frac{\sin^2\theta}{1 - \frac{\omega_{c,s}^2}{\omega^2}}.$$

For the electrons is $\frac{\omega_{ce}}{\omega} \gg 1$ and the second term of the sum is approximately $-\frac{\omega^2}{\omega_{ce}^2}\sin^2\theta$. In case of the ions, $\frac{\omega_{ci}}{\omega} \ll 1$, and the second term is $\approx \sin^2\theta$. Hence,

$$\omega^2 = \omega_{pi}^2 \underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + \omega_{pe}^2(\cos^2\theta - \frac{\omega^2}{\omega_{ce}^2}\sin^2\theta)$$

$$\omega^2\left[1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}\sin^2\theta\right] = \omega_{pi}^2\left[1 + \frac{m_i}{m_e}\cos^2\theta\right],$$

and finally

$$\omega^2 = \omega_{pi}^2\omega_{ce}^2\left[\frac{1 + \frac{m_i}{m_e}\cos^2\theta}{\omega_{ce}^2 + \omega_{pe}^2\sin^2\theta}\right].$$

Note that there is no dependence on $|\mathbf{k}|$. Now, for $\mathbf{k}\perp\mathbf{B}$ is $\sin\theta = 1$ and $\cos\theta = 0$, and using that

$$\omega_{pi}^2\omega_{ce}^2 = \omega_{pe}^2(\omega_{ce}\omega_{pi}),$$

one finds that

$$\omega^2 = (\omega_{ce}\omega_{pi})\left(\frac{\omega_{pc}^2}{\omega_{ce}^2 + \omega_{pe}^2}\right).$$

If now $\omega_{ce}^2 \ll \omega_{pe}^2$, then we will observe a *lower hybrid wave*

$$\boxed{\omega_{lh}^2 = \omega_{ce}\omega_{pe}.} \tag{4}$$