PHYS5150 — PLASMA PHYSICS

LECTURE 15 - MAGNETOHYDRODYNAMICS: CONVECTION AND DIFFUSION

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Spring 2023

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1 MAGNETEOHYDRODYNAMIC

2 CONVECTION AND DIFFUSION

In the previous section we derived the *generalized Ohm's law*. We now use Maxwell's equations to eliminate $\mathbf{j}$ from it

$$\mathbf{j} = \frac{1}{\eta}(\mathbf{E} + \bar{\mathbf{u}} \times \mathbf{B}) \quad \Big| \leftarrow \nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\nabla \times \mathbf{B} = \frac{\mu_0}{\eta}(\mathbf{E} + \bar{\mathbf{u}} \times \mathbf{B}) \quad \Big| \nabla \times$$

$$\nabla \times (\nabla \times \mathbf{B}) = \frac{\mu_0}{\eta}(\underbrace{\nabla \times \mathbf{E}}_{-\frac{\partial}{\partial t}\mathbf{B}} + \nabla \times (\bar{\mathbf{u}} \times \mathbf{B}))$$

$$\nabla(\underbrace{\nabla \cdot \mathbf{B}}_{0}) - \nabla^2 \mathbf{B} = \frac{\mu_0}{\eta}(-\frac{\partial}{\partial t}\mathbf{B} + \nabla \times (\bar{\mathbf{u}} \times \mathbf{B}))$$

and find that

$$\boxed{\frac{\partial}{\partial t}\mathbf{B} = \underbrace{\nabla \times (\bar{\mathbf{u}} \times \mathbf{B})}_{\text{Convection}} + \underbrace{\frac{\eta}{\mu_0}\nabla^2 \mathbf{B}}_{Diffusion}.} \tag{1}$$

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2.1 *Magnetic diffusion*

If the so-called *convection term* $\nabla \times (\bar{\mathbf{u}} \times \mathbf{B})$ in Eq. (1) can be ignored, then this equation takes the form of a *Diffusion equation* for the magnetic field $\mathbf{B}$

$$\frac{\partial \mathbf{B}}{\partial t} = D_m \nabla^2 \mathbf{B}$$

with the magnetic diffusion coefficient

$$D_m = (\mu_0 \sigma)^{-1}$$

and the conductivity $\sigma = \eta^{-1}$. Let us now characterize the magnitude of the magnetic diffusion. The typical diffusion time scale τ_d can be found by replacing $\nabla^2 B$ by B/L_B^2, where L_B is the characteristic gradient of the magnetic field. Then

$$\frac{\partial B}{\partial t} \approx D_m B / L_B^2$$

or

$$\frac{B}{\dot{B}} \approx \frac{L_B^2}{D_m},$$

and hence

$$B \sim \exp\left(\pm t/\tau_d\right),$$

where

$$\boxed{\tau_d = \frac{L_B^2}{D_m} = \mu_0 \sigma L_B^2.} \tag{2}$$

is the diffusion time scale.

2.1.1 Diffusion in solar wind

Let us consider the solar wind flowing from the Sun to Earth at a typical speed of $v_{sw} = 500\,\text{km/s}$, which takes about $\tau_{sw} \approx 3.5\,\text{d}$. Because the solar wind plasma is cold and collisions with neutrals are rare, we only have to consider Coulomb collisions, which implies for Eq. (2) that

$$\tau_d \approx 0.3 L_B^2.$$

What is the typical length scale of the magnetic field diffusion during this period of time? Setting τ_d to the solar wind travel time τ_{sw} we find that

$$\tau_d = \tau_{sw} = 3.5\,\text{d} \approx 0.3 L_B^2,$$

and thus

$$L_B \approx 1.9\sqrt{3.5\,\text{d}} \approx 10^3\,\text{m}.$$

The resulting diffusion length scale is much shorter than the distance between the Sun and Earth, which means that there is effectively no diffusion going on and the field is (practically) *frozen in* the solar wind plasma.

2.1.2 Diffusion in the Earth region

But the in the Earth region the situation is very different because of the prominence of collisions between the solar wind plasma particles with the neutrals of the Earth' atmosphere. Here $\sigma \sim 10^{-3}$ S/m and

$$\tau_d \approx 10^{-9} L_B^2.$$

Under such conditions, structures of tens of kilometers become diffusive within 1 second.

2.2 *Frozen-in magnetic flux*

If the conductivity approaches infinity or L_B becomes very large, Eq. (1) becomes

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\bar{\mathbf{u}} \times \mathbf{B}). \tag{3}$$

This equation is called the *hydromagnetic theorem* in analogy to the equation for the vorticity of non-viscous fluids. The theorem basically says that the field lines move with the plasma – the field appears to be *frozen-in the plasma*. The equation above is equivalent to

$$\frac{\partial}{\partial t}\mathbf{B} = -\nabla \times \mathbf{E} = \nabla \times (\bar{\mathbf{u}} \times \mathbf{B}),$$

and hence

$$0 = \nabla \times (\mathbf{E} + \bar{\mathbf{u}} \times \mathbf{B}).$$

The meaning of this identity is that the electric field disappears in frames co-moving with the plasma, or that electric fields can only result from Lorentz transformations.

2.3 *Magnetic Reynolds numbers*

We would like to have an easy means to verify whether the plasma is diffusive of convective. To this aim let us transform Eq. (1)

$$\frac{\partial}{\partial t}\mathbf{B} = \nabla \times (\bar{\mathbf{u}} \times \mathbf{B}) + \frac{\eta}{\mu_0}\nabla^2\mathbf{B}$$

into a dimensionless form

$$\frac{B}{\tau} = \frac{v \cdot B}{L_B} + \frac{B}{\tau_d}.$$

Here, v is the average plasma speed perpendicular to the field and L_B and τ_d are the characteristic length scale and diffusion time scale of the magnetic field, which we have introduced in 2.1.1. The ratio of the first and second term of the last expression

$$R_m = \frac{\frac{v \cdot B}{L_B}}{\frac{B}{\tau_d}} = \frac{v \cdot \tau_d}{L_B}$$

is the *magnetic Reynolds number*

$$\boxed{R_m = v L_B \mu_0 \sigma,}$$

which allows us to characterize the plasma state:

$$R_m \begin{array}{ll} \gg 1 & \text{flow dominated} \\ \ll 1 & \text{diffusion dominated.} \end{array}$$

We already mentioned that $R_m \gg 1$ implies that the magnetic field is frozen-in the plasma. To understand this strange claim better let us investigate the time dependence of the magnetic flux Φ encircled by a closed loop $c(t)$ around a bundle of frozen-in magnetic field lines:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \underbrace{\int_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{A}}_{\text{expl. time dependence of } \mathbf{B}} + \underbrace{\int_C \mathbf{B} \cdot (\mathbf{u} \times d\mathbf{l})}_{\text{extra flux enclosed as curve moves}}$$

$$= \int_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{A} + \int_C d\mathbf{l} \cdot (\mathbf{B} \times \mathbf{u}),$$

and after using Stoke's law

$$= \int_S \left\{ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) \right\} = 0.$$

The hydromagnetic theorem (3) implies that $\{\ldots\} = 0$, which means that the magnetic flux encircled by a closed loop is conserved even if the loop is moving at a relative speed. Bundles of magnetic field lines frozen into the plasma are often called *flux tubes*.