

PHYS5150 — PLASMA PHYSICS

LECTURE 13 - EQUILIBRIA

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1 STATIC EQUILIBRIA

As a first application we now consider static solutions, which means that $\frac{\partial}{\partial t} = 0$.

1.1 *Pressure gradient balanced by electric field*

Assume that nothing is moving, i.e. $\mathbf{u} = 0$ and that there is only a pressure gradient acting on the plasma, i.e. ∇p . The pressure gradient needs to be balanced by an electric field \mathbf{E} , i.e.

$$0 = -\nabla p - en\mathbf{E}.$$

With $\mathbf{E} = -\nabla\phi$ follows that

$$\nabla p = ne\nabla\phi.$$

Align pressure gradient with x and replace pressure by $n k_B T$

$$\frac{\partial}{\partial x}(n k_B T) = ne \frac{\partial}{\partial x}\phi.$$

The solution for that is the Boltzmann relation

$$n(x) = n_0 \exp\left(\frac{e\phi}{k_B T}\right).$$

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1.2 Pressure gradient balanced by $\mathbf{u} \times \mathbf{B}$

We now consider the situation where $\mathbf{u} \times \mathbf{B}$ is balanced by a pressure gradient, i.e.

$$\underbrace{(\mathbf{u} \cdot \nabla) \mathbf{u}}_0 = -\nabla p + nq(\mathbf{u} \times \mathbf{B}).$$

We assume that the left side of this equation is zero and will validate this assumption later. Then,

$$\begin{aligned} \nabla p &= nq(\mathbf{u} \times \mathbf{B}) = \mathbf{j} \times \mathbf{B} \quad \Big| \quad \mathbf{B} \times \\ \mathbf{B} \times \nabla p &= nq \mathbf{B} \times (\mathbf{u} \times \mathbf{B}) = nq \left[\mathbf{u}(\mathbf{B} \cdot \mathbf{B}) - \underbrace{\mathbf{B}(\mathbf{u} \cdot \mathbf{B})}_0 \right] = nq \mathbf{u} B^2, \end{aligned}$$

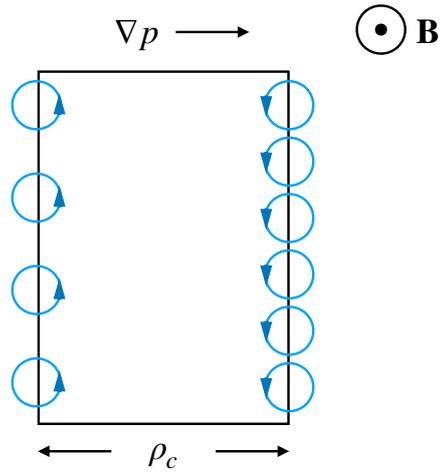
and after rearranging

$$\boxed{\mathbf{u}_D = \frac{\mathbf{B} \times \nabla p}{nqB^2}}. \quad (1)$$

This expression looks like a drift, and in fact this is one – the so-called *diamagnetic drift*. Note that \mathbf{u}_D is in direction perpendicular to \mathbf{B} . Before we discuss how a drift perpendicular to \mathbf{B} is even possible let's verify whether our initial assumption of $(\mathbf{u} \cdot \nabla) \mathbf{u} = 0$ is correct. In fact this turns out to be the case because $\mathbf{u}_D \perp \nabla p$.

Now, why is there a drift perpendicular to the magnetic field? Aren't the plasma particles orbiting the field lines? To understand this let's have a look how the pressure is balanced by the current inside a fluid element, which is ρ_c wide. Let the pressure gradient be oriented to the right. In this case, the number of particles gyrating along the field lines on the right side is larger than on the left side, and thus

$$dn = \Delta x \frac{dn}{dx} = \rho_c \frac{dn}{dx}.$$



In this configuration, the current is

$$j = (n_{down} - n_{up})qv_{\perp},$$

and after replacing with $(n_{down} - n_{up})$ with dn

$$\begin{aligned} j &= \rho_c \frac{dn}{dx} qv_{\perp} = \frac{mv_{\perp}}{|q|B} qv_{\perp} \frac{dn}{dx} \\ &= \frac{mv_{\perp}^2}{B} \frac{dn}{dx} = \frac{k_B T}{B} \nabla n. \end{aligned}$$

Now we use that $k_B T \nabla n = \nabla p$, and

$$j = \frac{\nabla p}{B}.$$

Given the original configuration this implies

$$\mathbf{j} \times \mathbf{B} = \nabla p.$$

1.3 Magnetic pressure

Let us return to the initial equation of the previous section

$$\nabla p = \mathbf{j} \times \mathbf{B},$$

which has lead us to the expression for the diamagnetic drift. This time we express the current density by Maxwell's equation

$$\mu_0 \mathbf{j} = \nabla \times \mathbf{B}.$$

Hence,

$$\begin{aligned}\nabla p &= \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}) \\ &= -\frac{1}{2\mu_0} \nabla \cdot (\mathbf{B} \cdot \mathbf{B}) + \frac{1}{\mu_0} \underbrace{(\mathbf{B} \cdot \nabla) \mathbf{B}}_0.\end{aligned}$$

The second term on the righthand side is more complicated than $\nabla \mathbf{B} = 0$, but $(\mathbf{B} \cdot \nabla) \mathbf{B} = 0$ is still true for straight field lines, and thus

$$\nabla p = -\nabla \left(\frac{B^2}{2\mu_0} \right)$$

or

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) = 0. \quad (2)$$

Eq. (2) implies that the sum of the kinetic pressure p and the magnetic pressure $\frac{B^2}{2\mu_0}$ is constant. The diamagnetic properties of a plasma are weakening an applied magnetic field resulting in a lower magnetic pressure inside the plasma, which needs to be balanced by an increase of the kinetic pressure.

The strength of the magnetic pressure relative to the kinetic pressure is often characterized by the dimensionless *plasma beta* parameter

$$\beta = \frac{p_{kin}}{p_{mag}} = \frac{2\mu_0 p}{B^2}. \quad (3)$$

A plasma is called a *low-beta plasma* when $\beta \ll 1$ and a *high-beta plasma* for $\beta \approx 1$ and greater.