PHYS5150 — PLASMA PHYSICS

LECTURE 12 - FLUID DESCRIPTION

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1 FLUID DESCRIPTION OF PLASMAS (CNTD.)

1.1 *Momentum*

We now derive the equation of motion for the fluid elements. We use that the seeked-after momentum equation is like a continuity equation for the momentum, i.e.

$$\frac{\partial}{\partial t}(nm\mathbf{u}) + \underbrace{\nabla(nm\mathbf{u}\otimes\mathbf{u})}_{strange} = \sum \frac{\text{forces}}{\text{volume}},$$

where

$$n \cdot m \cdot \mathbf{u} = \frac{\text{momentum}}{\text{volume}}$$

is the momentum density of a fluid element. While this approach is straight forward, the second term on the left side of the continuity equations demands further discussions. But before we come to this let us first have a closer look at the equation's right side, which accounts for external forces acting on the fluid element. Possible candidates for such forces are gravity

$$\frac{\mathbf{F}_g}{V} = n \cdot m \cdot \mathbf{g},$$

the electric force

$$\frac{\mathbf{F}_e}{V} = n \cdot q \cdot \mathbf{E},$$

and the magnetic forces acting on the electrons

$$\frac{\mathbf{F}_{m,e}}{V} = -n\,e\,\mathbf{u}_e \times \mathbf{B}$$

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and ions

$$\frac{\mathbf{F}_{m,i}}{V} = n\, e\, \mathbf{u}_e \times \mathbf{B}.$$

The latter two forces can be combined into a more handy single expression employing the current density $\mathbf{j}$

$$\frac{\mathbf{F}_{m,i}}{V} = n \cdot e((\mathbf{u}_i - \mathbf{u}_e) \times \mathbf{B}) = \mathbf{j} \times \mathbf{B}.$$

The forces we we have discussed so far are familiar from our discussion of single particle motion. There is, however, a force stemming from the pressure difference applied to a fluid element (remember that pressure is momentum flux!), which is unique for fluid mechanics. To see this, let us assume there is a differences between the pressure p_1 applied to the top face and p_2 applied to the bottom face of a fluid element. Then the rate of change of momentum through the volume element $\mathrm{d}x^3$ writes as

$$\frac{\partial}{\partial t}\left(n \cdot m \cdot u_z \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right) = [p_2 - p_1]\,\mathrm{d}x\,\mathrm{d}y \qquad \Bigg| \cdot \frac{1}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}$$

or after dividing by the volume

$$\frac{\partial}{\partial t}\left(n \cdot m \cdot u_z\right) = \frac{p_2 - p_1}{\mathrm{d}z} = \frac{p(z) - p(z + \mathrm{d}z)}{\mathrm{d}z} = -\frac{\partial p}{\partial z}.$$

We have already discussed that in fluid mechanics the pressure is not necessarily a just scalar, but often a tensor $\mathbf{P}$ to account for shear pressure terms. Including the $\nabla \mathbf{P}$ term the continuity equation for the momentum is

$$\frac{\partial}{\partial t}(nm\mathbf{u}) + \nabla(nm\mathbf{u} \otimes \mathbf{u}) = nm\mathbf{g} + qn(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \nabla \cdot \mathbf{P}.$$

Now let us try to make sense of the “strange” term $\nabla \cdot (n\mathbf{u} \otimes \mathbf{u})$. To this aim we drop for now the time dependence, ignore the external forces, and try to calculate the rate of accumulation of just the x-component of the momentum. Note that also the flux of x-momentum in y and z direction needs to be included, i.e.

$$\begin{aligned}
-\frac{\partial}{\partial t}(nmu_x)\hat{\mathbf{x}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = &\left[\mathrm{d}x\frac{\partial}{\partial x}(nmu_xu_x)\right]\hat{\mathbf{x}}\,\mathrm{d}y\,\mathrm{d}z + \\
&\left[\mathrm{d}y\frac{\partial}{\partial y}(nmu_xu_y)\right]\hat{\mathbf{x}}\,\mathrm{d}x\,\mathrm{d}z + \\
&\left[\mathrm{d}z\frac{\partial}{\partial z}(nmu_xu_z)\right]\hat{\mathbf{x}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Now we expand the $\frac{\partial}{\partial x_i}(\cdots)$ terms by employing the chain rule and get

$$\begin{aligned}
=&m\left[u_x\frac{\partial}{\partial x}(nu_x)+nu_x\frac{\partial}{\partial x}u_x\right]\hat{\mathbf{x}}\,\mathrm{d}x^3+\\
&m\left[u_x\frac{\partial}{\partial y}(nu_y)+nu_y\frac{\partial}{\partial y}u_x\right]\hat{\mathbf{x}}\,\mathrm{d}x^3+\\
&m\left[u_x\frac{\partial}{\partial z}(nu_z)+nu_z\frac{\partial}{\partial z}u_x\right]\hat{\mathbf{x}}\,\mathrm{d}x^3,
\end{aligned}$$

or

$$-\frac{\partial}{\partial t}(nmu_x)\hat{\mathbf{x}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = m\left[\nabla(n\mathbf{uu})\right]_x = m\left[\mathbf{u}\nabla(n\mathbf{u})+n(\mathbf{u}\nabla)\mathbf{u}\right]_x.$$

The y and z components of the momentum accumulation rate can be found similarly. After combining the three components into a single expression for $m\left(\nabla(n\mathbf{u}\otimes\mathbf{u})\right)$ we can write the momentum equation as

$$\frac{\partial}{\partial t}(nm\mathbf{u})+m\mathbf{u}(\nabla\cdot n\mathbf{u})+mn(\mathbf{u}\cdot\nabla)\mathbf{u}=\sum\frac{F}{V}.$$

We now resolve the time derivative on the left side

$$m\mathbf{u}\frac{\partial}{\partial t}(n)+mn\frac{\partial}{\partial t}\mathbf{u}+m\mathbf{u}(\nabla\cdot n\mathbf{u})+mn(\mathbf{u}\cdot\nabla)\mathbf{u}=\sum\frac{F}{V}$$

and rearrange the equation

$$\underbrace{m\mathbf{u}\frac{\partial}{\partial t}(n)+m\mathbf{u}(\nabla\cdot n\mathbf{u})}_{=\mathbf{u}Sm}+mn\frac{\partial}{\partial t}\mathbf{u}+mn(\mathbf{u}\cdot\nabla)\mathbf{u}-\sum\frac{F}{V}=0. \tag{1}$$

The first two terms are in fact forming the continuity equation multiplied by $m\mathbf{u}$

$$m\mathbf{u}\left(\frac{\partial}{\partial t}(n)+(\nabla\cdot n\mathbf{u})\right)=m\mathbf{u}S.$$

The appearance of the (mass) continuity equation in the one for the momentum is actually no surprise. $m\mathbf{u}S$ describes the change of the fluid's momentum due to gains or losses of plasma. If the plasma's mass density is conserved, $m\mathbf{u}S$ vanishes.

We now replace the first two terms in Eq. (1) by $\mathbf{u}Sm$ and obtain the equation of motion for charged fluids

$$\boxed{mn\left(\frac{\partial}{\partial t}+(\mathbf{u}\cdot\nabla)\right)\mathbf{u}=\sum\frac{F}{V}-\mathbf{u}Sm.} \tag{2}$$

1.1.1 Convective Derivative

In the momentum equation (2), the operator within the brackets is called the *convective derivative*. To understand this operator better, let us have a look at the time derivative of a fluid parameter G in a frame moving with the fluid in 1 dimension

$$\frac{\mathrm{D}G}{\mathrm{D}t} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}u_x.$$

The first term is the derivative of G fixed in space, while the second term describes the change of G due to the fluid moving to a new position. Similarily, in three dimensions the convective derivative $\frac{\mathrm{D}\mathbf{G}}{\mathrm{D}t}$ reads as

$$\frac{\mathrm{D}\mathbf{G}}{\mathrm{D}t} = \frac{\partial \mathbf{G}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{G}.$$

Note that in the last equation $(\mathbf{u}\cdot\nabla)$ is a *scalar* operator.

1.2 *Equation of state*

The plasma's equation of state ensures the conservation of energy and depends on the plasma properties. As an example let us consider a plasma which thermodynamic properties can be described by an ideal gas. We further assume that there are no sinks and sources. In this case, each particle has an average energy of $\frac{3}{2}\mathrm{k_B}T$, implying that the plasma's internal energy is

$$U = \frac{d}{2}N\mathrm{k_B}T,$$

where d is the plasma particle's number of degrees of freedom. When the plasma is doing work, U changes by

$$\mathrm{d}U = pA(-\,\mathrm{d}x) = -p\,\mathrm{d}V.$$

Using the ideal gas equation we get

$$pV = N\mathrm{k_B}T = \frac{2}{d}U,$$

and after differentiating

$$\begin{aligned}\mathrm{d}(pV) = p\,\mathrm{d}V + V\,\mathrm{d}p &\overset{!}{=} \frac{2}{d}\,\mathrm{d}U\\ &= -\frac{2}{d}p\,\mathrm{d}V.\end{aligned}$$

Rearranging the terms yields

$$0 = \frac{d+2}{d}p\,\mathrm{d}V + V\,\mathrm{d}p,$$

and after introducing the *adiabatic exponent* γ

$$\begin{aligned} 0 &= \gamma p \, \mathrm{d}V + V \, \mathrm{d}p \\ &= \gamma \frac{\mathrm{d}V}{V} + \frac{\mathrm{d}p}{p}. \end{aligned}$$

The solution of this equation above

$$\gamma \ln V + \ln p = \text{const.}$$

gives the *equation of state*

$$\boxed{p V^{\gamma} = \text{const.}}$$