PHYS5150 — PLASMA PHYSICS

LECTURE 11 -

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Spring 2023

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1 FLUID DESCRIPTION OF PLASMAS

1.1 *Fluid variables*

We first define the volume of a fluid parcel as

$$\mathrm{d}x^3 = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

which has the mass

$$m = (n_i m_i + n_e m_e)\,\mathrm{d}x^3$$

and the mass density

$$\rho = \frac{m}{\mathrm{d}x^3} = (n_i m_i + n_e m_e) = \rho_i + \rho_e.$$

Mass m and density ρ are fluid variables. We now introduce the average velocity $\mathbf{u}$ of the plasma particles in the fluid parcel. Obviously, the fluid parcel will flow at this speed. The distribution $f(\mathbf{v}|\mathbf{u})$ of particle speeds can often described by a shifted Maxwellian, i.e.

$$f(\mathbf{v}|\mathbf{u}) = \frac{n}{(\pi v_{th}^2)^{3/2}} \exp\left\{-\frac{(\mathbf{v}-\mathbf{u})^2}{v_{th}^2}\right\}.$$

Knowledge of $f(\mathbf{v})$ allows us to compute $\mathbf{u}$

$$\mathbf{u} = \frac{\int \mathbf{v} f(\mathbf{v})\,\mathrm{d}x^3}{\int f(\mathbf{v})\,\mathrm{d}x^3} = \frac{1}{n}\int \mathbf{v} f(\mathbf{v})\,\mathrm{d}x^3.$$

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The next fluid variable to discuss is the particle flux through a face of the fluid volume $\mathrm{d}x^3$. Let us consider the flux through the $\mathrm{d}x\,\mathrm{d}y$ face. Because flux is number of particles per area and time we can write

$$\Gamma_{xy} = \frac{N}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t},$$

or after multiplying with $\mathrm{d}z/\,\mathrm{d}z$

$$\Gamma_{xy} = \frac{N}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} = nu_z,$$

where u_z is the z component of fluid velocity. In general

$$\bar{\Gamma} = n\mathbf{u}.$$

The current density $\mathbf{j}$ of the fluid parcel is the charge flux

$$\mathbf{j} = nq\mathbf{u}.$$

The last fluid variable we need to consider is the pressure, which in fact is the flux of momentum evaluated in the frame moving with $\mathbf{u}$.

1.2 *Pressure*

In the case of a fluid pressure is flux of momentum evaluated in a frame moving with the fluid at velocity $\mathbf{u}$, i.e.

$$\mathbf{P} = \int \mathbf{v} \otimes \mathbf{p}(\mathbf{v}) f(\mathbf{v})\,\mathrm{d}\mathbf{v}.$$

In contrast to a gas pressure in a fluid is not necessarily a scalar, the general case is that the pressure is a tensor

$$P_{ij} = \begin{pmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{pmatrix}.$$

Let us consider an ideal gas. The momentum flux in x direction, i.e. P_{xx}, is

$$P_{xx} = \int v_x m v_x f(v)\,\mathrm{d}v = nm\langle\, v_x^2 \,\rangle = \tfrac{1}{2}k_B T = P_{yy} = P_{zz},$$

where the mixed tensor components vanish, e.g.

$$P_{xy} = \int v_x m v_y f(v)\,\mathrm{d}v = nm\langle\, v_x v_y \,\rangle = 0.$$

1.3 *Continuity equation*

$$\frac{\partial}{\partial t} m = \frac{\partial}{\partial t}(\rho \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z) = \sum \text{inward-flow} \; = \rho u_x \, \mathrm{d}y \, \mathrm{d}z$$

6 sides:

$$\int \frac{\partial \rho}{\partial t} \, \mathrm{d}x^3 = -\oint \rho \mathbf{u} \, \mathrm{d}A$$

divergence theorem: $\oint \mathbf{F} \, \mathrm{d}A = \int (\nabla \mathbf{F}) \, \mathrm{d}x^3$:

$$\int \frac{\partial \rho}{\partial t} \, \mathrm{d}x^3 = -\int \nabla(\rho \mathbf{u}) \, \mathrm{d}x^3$$

or

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) \right] \mathrm{d}x^3 = 0$$

Continuity equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0}$$

In general, if $\mathbf{H}$ is conserved, then

$$\frac{\partial}{\partial t} \mathbf{H} + \nabla(\mathbf{H} \otimes \mathbf{u}) = 0,$$

where $(\mathbf{H} \otimes \mathbf{u})$ is the flux of $\mathbf{H}$. If there is a source term $S = \frac{\text{new mass}}{\mathrm{d}x^3}$, then we need to change the continuity equation to

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = S.}$$