PHYS4150 — PLASMA PHYSICS

LECTURE 7 - ADIABATIC INVARIANTS

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Fall 2020

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1 ADIABATIC INVARIANTS

The presence of adiabatic invariants is actually a common phenomenon, which has been studied extensively in classical mechanics. Here we follow *Landau & Lifschitz* and consider a one-dimensional finite motion, where λ is a parameter describing a very slow change of the system. Here, slow means slow compared to the period T of the cyclic motion, i.e. $T\dot{\lambda} \ll \lambda$. Now, because λ is slowly changing, so is the energy E of the system, where $\dot{E} \sim \dot{\lambda}$. This implies that the change of energy is a function of λ, from what follows that there is a combination of E and λ, a so-called *adiabatic invariant*, which remains constant.

Now let $H(p,q;\lambda)$ be the Hamiltonian of such a system, where again λ is the parameter characterizing the slow change. Then,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\partial H}{\partial t} = \frac{\partial H}{\partial \lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}t}.$$

Now we average over one cycle T and assume that $\dot{\lambda}$ does not change on this time scale

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{\mathrm{d}\lambda}{\mathrm{d}t}\overline{\frac{\partial H}{\partial \lambda}}.$$

Now,

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^T \frac{\partial H}{\partial \lambda}\,\mathrm{d}t,$$

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and using that $\dot{q} = \frac{\partial H}{\partial p}$ we obtain

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T} \oint \frac{\partial H}{\partial \lambda} \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q.$$

By further noting that

$$T = \int_0^T \mathrm{d}t = \oint \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q$$

we get

$$\overline{\frac{\partial H}{\partial \lambda}} = \frac{\oint \frac{\partial H}{\partial \lambda} \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q}{\oint \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q},$$

and thus

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = \frac{\mathrm{d}\lambda}{\mathrm{d}t} \frac{\oint \frac{\partial H}{\partial \lambda} \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q}{\oint \left(\frac{\partial H}{\partial p} \right)^{-1} \mathrm{d}q}.$$

We have assumed that λ is constant along the integration path, which implies that $E = H(p,q;\lambda)$ is constant as well. Differentiating H with respect to λ gives

$$0 = \frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p} \frac{\partial p}{\partial \lambda},$$

and thus

$$\frac{\partial H}{\partial \lambda} \left(\frac{\partial H}{\partial p} \right)^{-1} = -\frac{\partial p}{\partial \lambda}.$$

After substituting this expression into our expression for the change of the mean energy we get

$$\overline{\frac{\mathrm{d}E}{\mathrm{d}t}} = -\frac{\mathrm{d}\lambda}{\mathrm{d}t} \frac{\oint \frac{\partial p}{\partial \lambda} \mathrm{d}q}{\oint \frac{\partial p}{\partial E} \mathrm{d}q},$$

or

$$0 = \oint \left(\frac{\partial p}{\partial E} \frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial p}{\partial \lambda} \frac{\mathrm{d}\lambda}{\mathrm{d}t} \right) \mathrm{d}q = \frac{\mathrm{d}}{\mathrm{d}t} \oint p \, \mathrm{d}q.$$

This result implies that the *adiabatic invariant*

$$\boxed{I = \frac{1}{2\pi} \oint p \, \mathrm{d}q} \tag{1}$$

remains constant even when the parameter λ is changing slowly. I is actually the area enclosed by periodic path of the system in the phase space.

1.1 *Example: Harmonic Oscillator*

As an example lets us consider a harmonic oscillator, which has the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2.$$

The system's path describes an ellipse with the semi-major axises $\sqrt{2mE}$ and $\sqrt{2E/m\omega^2}$, and the area

$$A = 2\pi\sqrt{2mE}\sqrt{2E/m\omega^2} = 2\pi\frac{E}{\omega}.$$

This implies that the oscillator has an adiabatic invariant

$$\boxed{I_{osc} = \frac{E}{\omega},} \tag{2}$$

which is conserved even when the oscillator's mass or k varies.