PHYS4150 — PLASMA PHYSICS

LECTURE 6 - SINGLE PARTICLE MOTION IN AN UNIFORM B FIELD

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Single particle motion in a uniform B field

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1 UNIFORM B AND E FIELDS

1.1 *E field parallel to B*

An E field parallel to $\mathbf{B}$ would only affect $\mathbf{v}_{\|}$ and result in a guiding center motion parallel to $\mathbf{B}$.

1.2 *E field perpendicular to B*

This case is more interesting than $\mathbf{E}\|\mathbf{B}$ and will lead us to new insights. Here, the equations of motion are

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\|} = 0$$

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\perp} = q\left(\mathbf{E}+\mathbf{v}_{\perp}\times\mathbf{B}\right).$$

We now transform the equations into an inertial frame moving at constant speed v_E perpendicular to $\mathbf{B}$. The fields in the new reference system are then

$$\mathbf{E}^{'} = \mathbf{E}+\mathbf{v}_E\times\mathbf{B}$$

$$\mathbf{B}^{'} = \mathbf{B},$$

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the velocity components are

$$\mathbf{v}_{\parallel}^{'} = \mathbf{v}_{\parallel}$$
$$\mathbf{v}_{\perp}^{'} = -\mathbf{v}_E + \mathbf{v}_{\perp},$$

and the new equation of motion for $\mathbf{v}_{\perp}$ is

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\perp}^{'} = q\left(\mathbf{E}^{'} + \mathbf{v}_{\perp}^{'} \times \mathbf{B}^{'}\right).$$

We now choose $\mathbf{v}_E$ such that $\mathbf{E}^{'}$ vanishes, i.e.

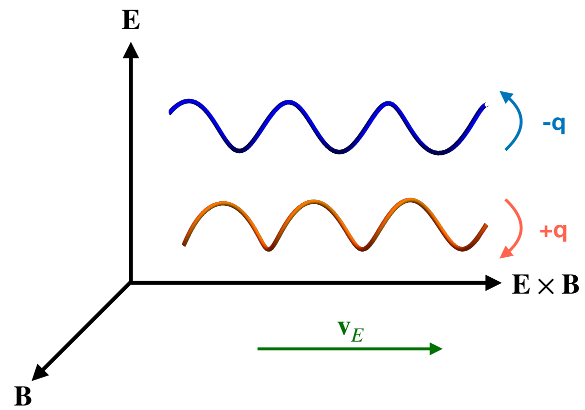
Remember that $(A \times B) \times C = (A \cdot C)B - (B \cdot C)A$

$$\mathbf{E}^{'} = \mathbf{E} + \mathbf{v}_E \times \mathbf{B} = 0 \quad \Big| \times \mathbf{B}$$
$$= \mathbf{E} \times \mathbf{B} + (\mathbf{v}_E \times \mathbf{B}) \times \mathbf{B} = \mathbf{E} \times \mathbf{B} + \cancel{(\mathbf{v}_E \cdot \mathbf{B})\mathbf{B}} - (\mathbf{B} \cdot \mathbf{B})\,\mathbf{v}_E,$$

and finally

$$\boxed{\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}}. \tag{1}$$

This is the velocity of the particle's guiding center drift caused by a uniform electric field perpendicular to **B**.



In the prime system the particle performs a simple gyromotion because its equation of motion is simply

$$m\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_{\perp}^{'} = q\left(\cancel{\mathbf{E}^{'}} + \mathbf{v}_{\perp}^{'} \times \mathbf{B}^{'}\right) = q\left(\mathbf{v}_{\perp}^{'} \times \mathbf{B}^{'}\right),$$

while in its initial system drifts the guiding center of the gyrating particle in $\mathbf{E} \times \mathbf{B}$ direction with the speed v_E.

The $\mathbf{E} \times \mathbf{B}$ drift has some remarkable properties. All particles *drift at the same speed* regardless of their charge, temperature, and mass. Furthermore, the drift motion *does not constitute a current*. Note also that in plasma physics the frame of rest is the one in which the electric field vanishes.

2 MOTION IN AN NONUNIFORM B FIELD

2.1 $\nabla\mathbf{B}$ *Drift*

We are interested in what happens when the particle moves in a nonuniform magnetic field. Lets assume that the B filed is aligned with the z axis, i.e. $\mathbf{B} = (0, 0, B_z)$, and that the density of the magnetic field lines increases in x direction, i.e. $\nabla\mathbf{B} \| \mathbf{x}$. We would expect a drift in y direction, because the radius of the gyromotion increases with decreasing magnetic field strength.

We now want to determine the average drift velocity due to the field gradient. Because the motion in x-direction is periodic

$$\oint F_x \, \mathrm{d}t = q \oint v_y B_z \, \mathrm{d}t = 0.$$

We assume the field gradient to be small, which allows us to expand B_z around its guiding center

$$B_z(x) \approx B_z(x_0) + \frac{\partial B_z}{\partial x}(x - x_0)$$

and get

$$\begin{aligned} 0 &= \oint \left\{ B_z(x_0) + \frac{\partial B_z}{\partial x}(x - x_0) \right\} v_y \, \mathrm{d}t \\ &= B_z(x_0) \underbrace{\oint v_y \, \mathrm{d}t}_{\Delta y} + \frac{\partial B_z}{\partial x} \oint (x - x_0) v_y \, \mathrm{d}t. \end{aligned}$$

We now make use of that

$$\oint (x - x_0) v_y \, \mathrm{d}t = \oint (x - x_0) \frac{\mathrm{d}y}{\mathrm{d}t} \, \mathrm{d}t = \oint (x - x_0) \, \mathrm{d}y$$

is approximately the area of a circle with radius ρ_c, and the second integral becomes

$$\oint (x - x_0) v_y \, \mathrm{d}t \approx -\frac{q}{|q|} \pi \rho_c^2.$$

Thus

$$\begin{aligned} 0 &= B_z \Delta y - \frac{\partial B_z}{\partial x} \frac{q}{|q|} \pi \rho_c^2 \\ \Delta y &= \frac{\frac{\partial B_z}{\partial x}}{B_z} \frac{q}{|q|} \pi \rho_c^2. \end{aligned}$$

During one gyration cycle $\Delta t = \frac{2\pi}{\omega_c}$ the particle drifts by Δy in y direction, which provides us with the drift speed

$$v_G = \frac{\Delta y}{\Delta t} = \frac{\partial B_z}{\partial x}\frac{1}{B_z}\frac{q}{|q|}\frac{1}{2}\omega_c\rho_c^2 = \frac{T_\perp}{qB_z}\left[\frac{1}{B_z}\frac{\partial B_z}{\partial x}\right],$$

where we have used that

$$T_\perp = \frac{m}{2}\omega_c^2\rho_c^2.$$

The general expression for the gradient B drift velocity is

$$\boxed{\mathbf{v}_G = \frac{T_\perp}{qB}\left[\frac{\hat{\mathbf{B}}\times\nabla\mathbf{B}}{B}\right].} \tag{2}$$

The direction of the grad B drift is in opposite direction for positive and negative charges and *causes therefore a current*.

2.2 *Curvature Drift*

A charged particle moving along a curved magnetic field line will experience a centrifugal force

$$F_C = m\frac{\mathbf{v}_\parallel^2}{R_C},$$

where R_C is the field curvature. This leads to a *curvature drift*

$$\mathbf{v}_C = -m\frac{\mathbf{v}_\parallel^2}{R_C^2}\left[\frac{\hat{\mathbf{R}}_C\times\hat{\mathbf{B}}}{qB^2}\right]$$

or after introducing the kinetic energy of the parallel motion $T_\parallel = \frac{1}{2}m\mathbf{v}_\parallel^2$

$$\boxed{\mathbf{v}_C = 2\frac{T_\parallel}{qB}\left[\frac{\hat{\mathbf{B}}\times\hat{\mathbf{R}}_C}{R_C}\right].} \tag{3}$$

The direction of the curvature drift is in opposite direction for positive and negative charges and *causes therefore a current*.