PHYS4150 — PLASMA PHYSICS

LECTURE 3 - PLASMA PROPERTIES: DEBYE SHIELDING

*Sascha Kempf**

G135, University of Colorado, Boulder

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Plasma properties: Debye shielding

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1 DEBYE SHIELDING

We now consider a *negative* test charge Q immersed in a homogeneous plasma. Q will attract ions but repellants electrons. The displacement of electrons produces a *polarization charge*, which shields the plasma from the test charge. The theory of shielding has been developed first in 1923 by Peter Debye and Erich Hückel for dielectric fluids.

To derive the shielding potential ϕ for the charge Q we assume a homogeneous plasma with electrons of temperature T_e and density n_e and a fixed background of ions of density n_0. After the test charge has established equilibrium with the plasma its potential is given by the Poisson equation

$$\nabla^2\phi(r) = -\frac{\rho}{\epsilon_0} = -\frac{e}{\epsilon_0}\left(n_0 - n_e(r)\right) \text{ with } \phi(\infty) = 0. \tag{1}$$

In an electrostatic field the velocity distribution of the electrons is

$$f_e(\mathbf{v}) = n_0\left\{\frac{m}{2\pi \mathrm{k_B}T}\right\}^{3/2}\exp\left\{-\frac{\frac{1}{2}m\mathbf{v}^2 + q\phi(r)}{\mathrm{k_B}T}\right\}.$$

The knowledge of $f_e(\mathbf{v})$ allows us to find the local electron number density $n_e(r)$

$$n_e(r) = \int_{\mathbb{R}} f_e(\mathbf{v})\,\mathrm{d}\mathbf{v} = n_0\exp\left\{\frac{e\phi(r)}{\mathrm{k_B}T}\right\},$$

electrons: $q = -e$

*sascha.kempf@colorado.edu

which we substitute into Eq. (1)

$$\nabla^2\phi = -\frac{e}{\epsilon_0}n_0\left(1-\exp\left\{\frac{e\phi}{\mathrm{k_B}T}\right\}\right).$$

We expand the exponential term into a Taylor series to linearize the quation for ϕ

$$\exp\left\{\frac{e\phi}{\mathrm{k_B}T}\right\} = 1+\frac{e\phi}{\mathrm{k_B}T}+\frac{1}{2}\left(\frac{e\phi}{\mathrm{k_B}T}\right)^2+\frac{1}{3!}\left(\frac{e\phi}{\mathrm{k_B}T}\right)^3+\cdots$$

and keep only the first two terms

$$\nabla^2\phi \approx \frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T}.$$

Because the plasma is isotropic we now want to make use of the spherical symmetry of the problem. To this aim we express the Laplace operator in spherical coordinates

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right)+\frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi)+\frac{1}{r^2\sin^2\theta}\partial^2_\phi\phi$$

and drop the symmetric angular terms

$$\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) = \frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T}.$$

This leads to an ordinary second order linear differential equation

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right)-\frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T} = 0$$

$$\frac{1}{r}\partial_r^2(r\phi)-\frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T} = 0$$

$$\partial_r^2(r\phi)-\frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T}(r\phi) = y''-\frac{n_0}{\epsilon_0}\frac{e^2\phi}{\mathrm{k_B}T}y = 0 \text{ with } y=(r\phi).$$

The solutions of $y''+a^2y=0$ have the general form

$$y(x) = \frac{c}{x}\exp(\pm ax),$$

from which follows that

$$\phi(r) = \frac{A}{r}\exp\left(-\frac{r}{\lambda_D}\right)$$

with

$$\boxed{\lambda_D^2 = \frac{\epsilon_0\mathrm{k_B}T_e}{n_0e^2}} \tag{2}$$

being the *Debye length*. The value for the constant A can be found by using the fact that at large distances $\phi(r)$ must asymptotically approach *Coulomb's law* and we yield

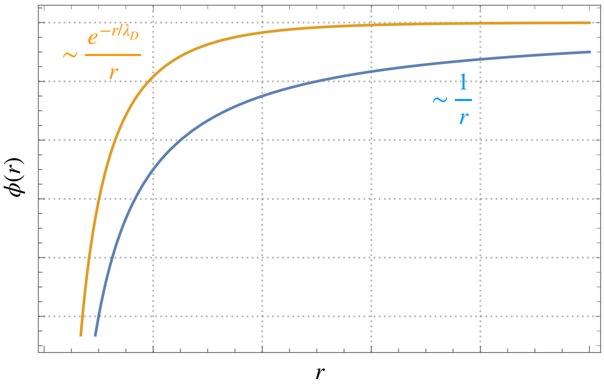

Figure 1: Comparison between the Debye-Hückel potential (orange) of a charge immersed in a plasma and the Coulomb potential (blue) of a free charge.

the so-called *Debye-Hückel potential*

$$\boxed{\phi(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\exp\left(-\frac{r}{\lambda_D}\right)} \tag{3}$$

(Fig. 1). A useful relation for the Debye length is

$$\lambda_D = 7430\text{m}\sqrt{\frac{T}{e\text{V}}\frac{\text{m}^{-3}}{n}}. \tag{4}$$