PHYS4150 — PLASMA PHYSICS

LECTURE 20 - ELECTROSTATIC WAVES IN COLD MAGNETIZED PLASMAS

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1 ELECTROSTATIC WAVES IN COLD MAGNETIZED PLASMAS

- Why cold? Initial interest was for ionospheric science
- Consider both, electron and ion motion
- Equations:
  - Poisson equation
  - Continuity equation
  - 3-D momentum equation with $p = 0$ (cold ionosphere) and $B_0 \neq 0$
- Geometry: Wave might have $\mathbf{k}||\mathbf{B}$, $\mathbf{k}\perp\mathbf{B}$, or in-between

Without loss of generality we chose $\mathbf{B}$ and $\mathbf{k}$ such that

$$\mathbf{B} = [0, 0, B_z]$$

$$\mathbf{k} = [k_x, 0, k_z].$$

We consider electrostatic waves, which implies that $\mathbf{k}||\mathbf{E}$, and consequently

$$\mathbf{E}(\mathbf{x}, t) = [E_x, 0, E_z]e^{i(k_x x + k_z z - \omega t)}.$$

We note further that

$$\mathbf{E} = -\nabla\phi = [-ik_x\phi, 0, -ik_z\phi]$$

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and

$$\nabla \cdot \mathbf{E} = -\nabla^2 \phi = k_x^2 \phi + k_z^2 \phi.$$

The Poisson equation gives

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0} = -\frac{1}{\epsilon_0} \sum_{s=i,e} \delta n_s q_s,$$

the continuity equation reads

$$\begin{aligned} \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) &= 0 \\ -i\omega \delta n_s + n_0 i(k_x u_x + k_z u_z) &= 0, \end{aligned}$$

and the momentum equation writes as

$$m_s \cancel{n_s} \left(\frac{\partial \mathbf{u}_s}{\partial t} + \underbrace{(\mathbf{u}_s \cdot \nabla)\mathbf{u}_s}_{O(\mathbf{u}_s^2)\sim 0} \right) = \underbrace{-\nabla p}_{T=0\to 0} + \cancel{n_s} q_s \mathbf{E} + \cancel{n_s} q_s \mathbf{u}_s \times \mathbf{B}.$$

We first solve the $\hat{\mathbf{x}}$ component of the momentum equation to find $u_{x,s}$ as function of $u_{y,s}$:

$$-i\omega m_s u_{x,s} = q_s E_x + q_s u_{y,s} B_z,$$

then solve then the $\hat{\mathbf{y}}$ component to obtain a second relation for these velocities

$$\begin{aligned} -i\omega m_s u_{y,s} &= -q_s u_{x,s} B_z \\ u_{y,s} &= -\frac{i q_s u_{x,s} B_z}{\omega m_s} = -i \frac{\omega_{c,s}}{\omega} u_{x,s}, \end{aligned}$$

and subsititute $u_{y,s}$ into the $\hat{\mathbf{x}}$ component

$$\begin{aligned} -i\omega m_s u_{x,s} &= q_s E_x + \underbrace{q_s B_z}_{m_s \omega_{ccs}} u_{y,s} \quad \Big| \leftarrow u_{y,s} \\ &= q_s E_x - i \frac{m_s}{\omega} \omega_{c,s}^2 u_{x,s} \\ -i\omega^2 u_{x,s} &= \frac{q_s}{m_s} E_x \omega - i\omega_{c,s}^2 u_{x,s} \\ u_{x,s} &= i \frac{q_s}{m_s} E_x \left(\frac{\omega}{\omega^2 - \omega_{c,s}^2} \right). \end{aligned}$$

Finally, we obtain the relation for $u_{z,s}$ from the $\hat{\mathbf{z}}$ component:

$$-i\omega m_s u_{z,s} = q_s E_z$$
$$u_{z,s} = i\frac{q_s}{m_s}E_z\frac{1}{\omega}.$$

We now put velocities into the continuity equation and yield

$$0 = -i\omega\delta n_s + i n_0\left(i\frac{q_s}{m_s}k_x E_x\frac{\omega}{\omega^2-\omega_{c,s}^2} + i\frac{q_s}{m_s}k_z E_z\frac{1}{\omega}\right).$$

We use that $\mathbf{E} = -\nabla\phi$, implying that $E_x = -ik_x\phi$ and $E_z = -ik_z\phi$. After rearranging the equation above we get an expression for δn_s

$$\delta n_s = n_0\frac{q_s}{m_s}\left[\frac{k_x^2}{\omega^2-\omega_{c,s}^2}+\frac{k_z^2}{\omega^2}\right]\delta\phi,$$

which we put into the Poisson equation

$$(k_x^2+k_z^2)\delta\phi = \frac{1}{\epsilon_0}\sum_{s=i,e}\delta n_s q_s$$
$$= \delta\phi\sum_{s=i,e}\underbrace{\left(\frac{n_0 q_s^2}{\epsilon_0 m_s}\right)}_{\omega_{p,s}}\left[\frac{k_x^2}{\omega^2-\omega_{c,s}^2}+\frac{k_z^2}{\omega^2}\right] \quad \Bigg| \cdot\frac{\omega^2}{k^2},$$

and after rearranging the equation for ω we get

$$\boxed{\omega^2 = \sum_{s=i,e}\frac{\omega_{p,s}^2}{k^2}\left[k_z^2+\frac{k_x^2}{1-\frac{\omega_{c,s}^2}{\omega^2}}\right]} \tag{1}$$

The resulting *dispersion relation for electrostatic waves propagating in a cold plasma* is rather complicated, but very useful, albeit only in simplified limits.

1.1 *Dispersion relation for* $\mathbf{B}\to 0$

This case implies that $\omega_{c,s}\to 0$, which recovers the *plasma oscillation case*

$$\boxed{\omega^2 = \omega_{pe}^2+\omega_{pi}^2.} \tag{2}$$

1.2 *Dispersion relation for* $\mathbf{k}\|\mathbf{B}$

In this case there is no $\delta u\times B_z$ force acting on the plasma particles. Thus, this scenario is similar to the previous one and we get again

$$\boxed{\omega^2 = \omega_{pe}^2 + \omega_{pi}^2.} \tag{3}$$

1.3 *Strongly magnetized plasma*

Strongly magnetized means that $\mathbf{B} \to \infty$, and therefore is $\omega_{c,s} \gg \omega$,

$$\frac{k_x^2}{1 - \frac{\omega_{c,s}^2}{\omega^2}} \sim O\left(\frac{\omega}{\omega_{c,s}}\right)^2 \approx 0,$$

and the dispersion relation becomes

$$\omega^2 = (\omega_{pe}^2 + \omega_{pi}^2)\frac{k_z^2}{k^2}.$$

After replacing $\frac{k_z^2}{k^2}$ by $\cos\theta$, where θ is the angle between $\mathbf{B}$ and $\mathbf{k}$, the dispersion relation for a strongly magnetized plasma writes as

$$\boxed{\omega^2 = (\omega_{pe}^2 + \omega_{pi}^2)\cos\theta.} \tag{4}$$