PHYS4150 — PLASMA PHYSICS

LECTURE 14 - MAGNETOHYDRODYNAMIC

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Fall 2020

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1 MAGNETEOHYDRODYNAMIC

1.1 *Parameters*

Mass density:

$$\rho \equiv n_e m_e + n_i m_i \approx n(n_e + n_i)$$

Fluid velocity:

$$\bar{\mathbf{u}} \equiv \frac{1}{\rho}(n_e m_e \mathbf{u}_e + n_i m_i \mathbf{u}_i) \approx \frac{m_e \mathbf{u}_e + m_i \mathbf{u}_i}{m_e + m_i}$$

Current density:

$$\mathbf{j} \equiv e(m_i \mathbf{u}_i - n_e \mathbf{u}_e) \approx n\,e(\mathbf{u}_i - \mathbf{u}_e)$$

1.2 *Continuity equation*

$$\frac{\partial n_i}{\partial t} + \nabla(n_i \mathbf{u_i}) = 0 \qquad \Big| \cdot m_i$$

$$\frac{\partial n_e}{\partial t} + \nabla(n_e \mathbf{u_e}) = 0 \qquad \Big| \cdot m_e$$

Add

$$\frac{\partial}{\partial t} \underbrace{(n_e m_e + n_i m_i)}_{\rho} + \nabla \underbrace{(n_e m_e \mathbf{u_e} + n_i m_i \mathbf{u_i})}_{\rho \bar{\mathbf{u}}} = 0$$

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$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{\mathbf{u}}) = 0.}$$

1.3 *Momentum equation*

Momentum equations for electron and ion fluids:

$$\frac{\partial}{\partial t} m_e n_e \mathbf{u}_e + \underbrace{m_e n_e (\mathbf{u}_e \cdot \nabla) \mathbf{u}_e}_{\approx 0} = -e n_e m_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + n_e m_e \mathbf{g} - \nabla p_e + P_{ei} \quad (1)$$

$$\frac{\partial}{\partial t} m_i n_i \mathbf{u}_i + \underbrace{m_i n_i (\mathbf{u}_i \cdot \nabla) \mathbf{u}_i}_{\approx 0} = e n_i m_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + n_i m_i \mathbf{g} - \nabla p_i + P_{ie} \quad (2)$$

Gravity is just a placeholder for any non-magnetic force. $P_{ei} = -P_{ie}$ describes the friction between the fluids. Add Eqs. (1) and (2) gives

$$n \frac{\partial}{\partial t} (m_i \mathbf{u}_i + m_e \mathbf{u}_e) = e n (\mathbf{u}_i - \mathbf{u}_e) \times \mathbf{B} - \nabla p + n (m_i + m_e) \mathbf{g}. \quad (3)$$

The electric field cancels due to quasi neutrality. The resulting MHD momentum equation is then

$$\boxed{\rho \frac{\partial \bar{\mathbf{u}}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho \mathbf{g}.}$$

We have lost any dependence on the friction term, which we need to recover. Obviously, we need another equation.

1.4 *Generalized Ohm's law*

We start again with the fluid momentum equations,

$$\frac{\partial}{\partial t} m_i n_i \mathbf{u}_i = + e n_i m_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) + n_i m_i \mathbf{g} - \nabla p_i + P_{ie} \quad \Big| \cdot m_e$$

$$\frac{\partial}{\partial t} m_e n_e \mathbf{u}_e = - e n_e m_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + n_e m_e \mathbf{g} - \nabla p_e + P_{ei} \quad \Big| \cdot m_i$$

but this time we subtract them

$$\begin{aligned} m_i m_e n \frac{\partial}{\partial t} (\mathbf{u}_i - \mathbf{u}_e) =& e n (m_e + m_i) \mathbf{E} + e n (m_e \mathbf{u}_i + m_i \mathbf{u}_e) \times \mathbf{B} \\ & - m_e \nabla p_i + m_i \nabla p_e - (m_e + m_i) P_{ei} \end{aligned}$$

We will derive an expression for P_{ie} later of this semester. For now we note that

$$P_{ei} \sim \text{Coulomb force } \sim e^2$$
$$P_{ei} \sim n_e \text{ and } n_i \sim n^2$$
$$P_{ei} \sim \text{relative velocities } \sim (\mathbf{u}_i - \mathbf{u}_e)$$

and thus

$$P_{ei} = \eta e^2 n^2 (\mathbf{u}_i - \mathbf{u}_e),$$

where the proportionality constant is the *resistivity*. Now

$$\begin{aligned}\frac{m_i m_e n}{e}\frac{\partial}{\partial t}\left(\frac{\mathbf{j}}{n}\right) =& e\,\rho\mathbf{E} + e\,n(m_i\mathbf{u}_e + m_e\mathbf{u}_i)\times\mathbf{B}\\ &- m_e\nabla p_i + m_i\nabla p_e - (m_e+m_i)\eta e\,n\,\mathbf{j}\end{aligned}$$

Using that

$$\begin{aligned}m_e\mathbf{u}_i + m_i\mathbf{u}_e &= m_i\mathbf{u}_i + m_e\mathbf{u}_e + m_i(\mathbf{u}_e - \mathbf{u}_i) + m_e(\mathbf{u}_i - \mathbf{u}_e)\\ &= \frac{\rho}{n}\bar{\mathbf{u}} - (m_i - m_e)\frac{\mathbf{j}}{n\,e}\end{aligned}$$

and hence

$$\begin{aligned}\frac{m_i m_e n}{e}\frac{\partial}{\partial t}\left(\frac{\mathbf{j}}{n}\right) =& e\,\rho\mathbf{E} + e\,\rho\,\bar{\mathbf{u}}\times\mathbf{B} - (m_i - m_e)\mathbf{j}\times\mathbf{B}\\ &- m_e\nabla p_i + m_i\nabla p_e - \rho\,e\,\eta\mathbf{j}\end{aligned}$$

After dividing by ρe and rearranging terms

$$\mathbf{E} + \bar{\mathbf{u}}\times\mathbf{B} - \eta\mathbf{j} = \frac{1}{e\rho}\left\{\frac{m_i m_e n}{e}\frac{\partial}{\partial t}\left(\frac{\mathbf{j}}{n}\right) + (m_i - m_e)\mathbf{j}\times\mathbf{B} + m_e\nabla p_i - m_i\nabla p_e\right\}$$

For MHD approximation we assume slow enough motions for $\frac{\partial}{\partial t}$ to be neglected. Slow enough means slower than ω_c^{-1}. We also take the limit $m_i \gg m_i$ and get the *generalized Ohm's law*

$$\boxed{\mathbf{E} + \bar{\mathbf{u}}\times\mathbf{B} = \eta\mathbf{j} + \frac{1}{e\,n}(\mathbf{j}\times\mathbf{B} - \nabla p_e)\,.}$$

For many plasma's the term in the brackets can be neglected

$$\boxed{\mathbf{E} + \bar{\mathbf{u}}\times\mathbf{B} = \eta\mathbf{j}.}$$

The case of $\eta = 0$ is called *ideal MHD*.

1.5 *The Magnetohydrodynamics (MHD) equations*

Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{\mathbf{u}}) = 0 \tag{4}$$

Pressure:

$$pV^{\gamma} = \text{const.} \tag{5}$$

Momentum equation

$$\rho \frac{\partial \bar{\mathbf{u}}}{\partial t} = \mathbf{j} \times \mathbf{B} - \nabla p + \rho \mathbf{g} \tag{6}$$

Generalized Ohm's law

$$\mathbf{E} + \bar{\mathbf{u}} \times \mathbf{B} = \eta \mathbf{j} \tag{7}$$

Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial t}{\partial \mathbf{B}} \tag{8}$$

Ampere's law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \tag{9}$$

Gauss' law

$$\nabla \cdot \mathbf{B} = 0 \tag{10}$$