PHYS4150 — PLASMA PHYSICS

LECTURE 13 -

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Fall 2020

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1 FLUID DESCRIPTION OF PLASMAS (CNTD.)

1.1 *Equation of state*

The plasma's equation of state ensures the conservation of energy and depends on the plasma properties. As an example let us consider a plasma which thermodynamic properties can be described by an ideal gas. We further assume that there are no sinks and sources. In this case, each particle has an average energy of $\frac{3}{2}\mathrm{k_B}T$, implying that the plasma's internal energy is

$$U = \frac{d}{2} N \mathrm{k_B} T,$$

where d is the plasma particle's number of degrees of freedom. When the plasma is doing work, U changes by

$$\mathrm{d}U = pA(-\,\mathrm{d}x) = -p\,\mathrm{d}V.$$

Using the ideal gas equation we get

$$pV = N\mathrm{k_B}T = \frac{2}{d}U,$$

and after differentiating

$$\mathrm{d}(pV) = p\,\mathrm{d}V + V\,\mathrm{d}p \stackrel{!}{=} \frac{2}{d}\,\mathrm{d}U$$
$$= -\frac{2}{d}p\,\mathrm{d}V.$$

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Rearranging the terms yields

$$0 = \frac{d+2}{d} p\,\mathrm{d}V + V\,\mathrm{d}p,$$

and after introducing the *adiabatic exponent* γ

$$\begin{aligned} 0 &= \gamma p\,\mathrm{d}V + V\,\mathrm{d}p \\ &= \gamma \frac{\mathrm{d}V}{V} + \frac{\mathrm{d}p}{p}. \end{aligned}$$

The solution of this equation above

$$\gamma \ln V + \ln p = \text{const.}$$

gives the *equation of state*

$$\boxed{pV^\gamma = \text{const.}}$$

2 STATIC EQUILIBRIA

As a first application we now consider static solutions, which means that $\frac{\partial}{\partial t} = 0$.

2.1 *Pressure gradient balanced by electric field*

Assume that nothing is moving, i.e. $\mathbf{u} = 0$ and that there is only a pressure gradient acting on the plasma, i.e. $\sum \mathbf{F}$. The pressure gradient needs to be balanced by an electric field $\mathbf{E}$, i.e.

$$0 = -\nabla p - en\mathbf{E}.$$

With $\mathbf{E} = -\nabla\phi$ follows that

$$\nabla p = ne\nabla\phi.$$

Align pressure gradient with x and replace pressure by $n\mathrm{k_B}T$

$$\frac{\partial}{\partial x}(n\mathrm{k_B}T) = ne\frac{\partial}{\partial x}\phi.$$

The solution for that is the Boltzmann relation

$$n(x) = n_0 \exp\left(\frac{e\phi}{\mathrm{k_B}T}\right).$$

2.2 *Pressure gradient balanced by* $\mathbf{u} \times \mathbf{B}$

We now consider the situation where $\mathbf{u} \times \mathbf{B}$ is balanced by a pressure gradient, i.e.

$$\underbrace{(\mathbf{u}\cdot\nabla)\,\mathbf{u}}_{0} = -\nabla p + nq\,(\mathbf{u}\times\mathbf{B}).$$

We assume that the left side of this equation is zero and will validate this assumption later. Then,

$$\nabla p = nq(\mathbf{u}\times\mathbf{B}) = \mathbf{j}\times\mathbf{B} \quad \Big|\mathbf{B}\times$$

$$\mathbf{B}\times\nabla p = nq\mathbf{B}\times(\mathbf{u}\times\mathbf{B}) = nq\left[\mathbf{u}(\mathbf{B}\cdot\mathbf{B}) - \mathbf{B}\underbrace{(\mathbf{u}\cdot\mathbf{B})}_{0}\right] = nq\mathbf{u}B^2,$$

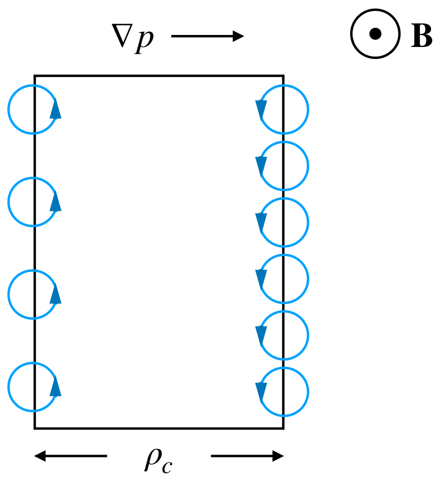
and after rearranging

$$\boxed{\mathbf{u}_D = \frac{\mathbf{B}\times\nabla p}{nqB^2}.} \tag{1}$$

This expression looks like a drift, and in fact this is one – the so-called *diamagnetic drift*. Note that $\mathbf{u}_D$ is in direction perpendicular to $\mathbf{B}$. Before we discuss how a drift perpendicular to $\mathbf{B}$ is even possible let's verify whether our initial assumption of $(\mathbf{u}\cdot\nabla)\,\mathbf{u} = 0$ is correct. In fact this turns out to be the case because $\mathbf{u}_D \perp \nabla p$.

Now, why is there a drift perpendicular to the magnetic field? Aren't the plasma particles orbiting the field lines? To understand this let's have a look how the pressure is balanced by the current inside a fluid element, which is ρ_c wide. Let the pressure gradient be oriented to the right. In this case, the number of particles gyrating along the field lines on the right side is larger than on the left side, and thus

$$\mathrm{d}n = \Delta x\frac{\mathrm{d}n}{\mathrm{d}x} = \rho_c\frac{\mathrm{d}n}{\mathrm{d}x}.$$



In this configuration, the current is

$$j = (n_{down} - n_{up})qv_\perp,$$

and after replacing with $(n_{down} - n_{up})$ with $\mathrm{d}n$

$$j = \rho_c \frac{\mathrm{d}n}{\mathrm{d}x} q v_\perp = \frac{m v_\perp}{|q| B} q v_\perp \frac{\mathrm{d}n}{\mathrm{d}x}$$
$$= \frac{m v_\perp}{B} \frac{\mathrm{d}n}{\mathrm{d}x} = \frac{\mathrm{k_B} T}{B} \nabla n.$$

Now we use that $\mathrm{k_B} T \nabla n = \nabla p$, and

$$j = \frac{\nabla p}{B}.$$

Given the original configuration this implies

$$\mathrm{j} \times \mathbf{B} = \nabla p.$$

2.3 *Magnetic pressure*

Let us return to the initial equation of the previous section

$$\nabla p = \mathbf{j} \times \mathbf{B}.,$$

which has lead us to the expression for the diamagnetic drift. This time we express the current density by Maxwell's equation

$$\mu_0 \mathbf{j} = \nabla \times \mathbf{B}.$$

Hence,

$$\nabla p = \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B})$$
$$= -\nabla \cdot (\mathbf{B} \cdot \mathbf{B}) + \underbrace{(\mathbf{B} \cdot \nabla) \mathbf{B}}_{0}.$$

The second term on the righthand side is more complicated than $\nabla \mathbf{B} = 0$, but $(\mathbf{B} \cdot \nabla) \mathbf{B} = 0$ is still true for straight field lines, and thus

$$\nabla p = -\nabla \left(\frac{B^2}{2\mu_0} \right)$$

or

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) = 0. \tag{2}$$

Eq. (2) implies that the sum of the kinetic pressure p and the magnetic pressure $\frac{B^2}{2\mu_0}$ is constant. The diamagnetic properties of a plasma are weakening an applied magnetic field resulting in a lower magnetic pressure inside the plasma, which needs to be balanced by an increase of the kinetic pressure.

2.4 *Example: The Langmuir-Child law*

emitted electron current: $j = n\,q\,u =$ constant, i.e. $\nabla n\mathbf{u} = 0$.

emission velocity: u_0

emission energy: $\frac{m}{2}u_0^2$

momentum equation:

$$nmu\frac{\mathrm{d}u}{\mathrm{d}x} = nqE = -neE$$
$$\frac{m}{2}\frac{\mathrm{d}u^2}{\mathrm{d}x} = -e\frac{\mathrm{d}\phi}{\mathrm{d}dx}$$
$$\frac{mu^2}{2} - e\phi = \text{constant} = \frac{mu_0^2}{2}$$
$$u^2 = u_0^2 + \frac{2e\phi}{m}.$$

but

$$n = \frac{-j}{eu} = -\frac{j}{e\sqrt{u_0^2 + \frac{2e\phi}{m}}}.$$

Also

$$\nabla E = \frac{\mathrm{d}E}{\mathrm{d}x} = -\frac{ne}{\epsilon_0} = \frac{j}{\epsilon_0\sqrt{u_0^2 + \frac{2e\phi}{m}}}.$$

Now we have 2 first oder differential equations:

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = -E$$
$$\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{j}{\epsilon_0\sqrt{u_0^2 + \frac{2e\phi}{m}}} = -\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}$$